Name: _____

Monday 12th September   Time: ~~45~~ 50 minutes   Mark   ~~/40~~ 45

Section 1 – Calculator free 20 marks

1.   [5 marks – 4 and 1]

The noise level, in decibels, of the Year 9 class next door is increasing at a rate proportional to the square root of itself, i.e. $\dfrac{dN}{dt} \propto \sqrt{N}$. The noise level started at 64dB and rose to 100dB in 20 minutes.

(a) Write and solve an appropriate differential equation to model this situation

$$\frac{dN}{dt} = kN^{\frac{1}{2}}$$

$$\therefore \int \frac{dN}{N^{\frac{1}{2}}} = \int k \, dt$$

$$\therefore 2\sqrt{N} = kt + c$$

$(0, 64) \Rightarrow c = 16$

$(20, 100) \Rightarrow 20 = 20k + 16$

$$k = \frac{1}{5}$$

ie. $2\sqrt{N} = \dfrac{t}{5} + 16$

$$\sqrt{N} = \frac{t}{10} + 8$$

$$N = \left(\frac{t}{10} + 8\right)^2$$

(b) How long from the start will it take to exceed the pain threshold by reaching 144 dB?

$$\frac{t}{10} + 8 = 12$$

$$t = 40 \text{ minutes}$$

2. [6 marks – 1 each]

A particle is moving in simple harmonic motion with its acceleration at time t given by

$\dfrac{d^2x}{dt^2} = 4\cos(kt)$, for k a constant.

(a) Express each of these quantities in terms of k:

(i)   the period of motion

$\dfrac{2\pi}{k}$

(ii)   the frequency of motion

$\dfrac{k}{2\pi}$

(iii) the amplitude of the motion

$\dfrac{4}{k^2}$

(iv)   the displacement $x(t)$

$x(t) = -\dfrac{4}{k^2}\cos(kt)$

(b) If the maximum speed of the particle is 6 units, evaluate:

(v) k

$\dfrac{4}{k} = 6 \quad \Rightarrow \quad k = \dfrac{2}{3}$

(vi)   the amplitude

$A = \dfrac{4}{\frac{4}{9}} = 9$

3. [3 marks]

An object, with displacement x and velocity v, moves so that $v = 3x - 5$ m/s.

What is the acceleration of this object it is 2 metres from the origin?

$a = \dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right) = \dfrac{d}{dx}\left(\tfrac{1}{2}[3x-5]^2\right)$

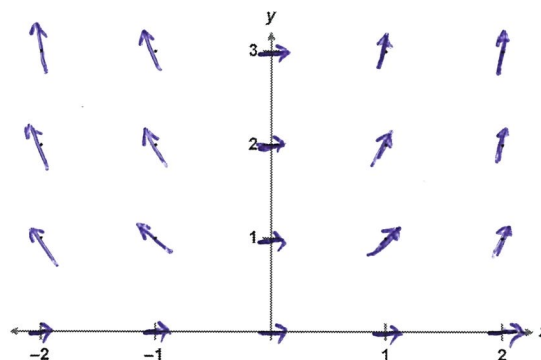$= \tfrac{1}{\cancel{2}} \cdot \cancel{2} \cdot (3x-5) \cdot 3$

$a(x=2) = 3 \ \text{m/sec}^2$

4.   [6 marks – 1, 2 and 3]

(a)   Enter the values of $\dfrac{dy}{dx} = xy$ in this table.

x	y	$\dfrac{dy}{dx}$
1	2	2
−2	3	−6
3	0	0

(b)   Use these values and others from the 20 integer points marked to draw the slope field for the differential equation $\dfrac{dy}{dx} = xy$.

(c)   Solve the differential equation $\dfrac{dy}{dx} = xy,\; y > 0$ if $y = 2$ when $x = 0$.

$$\int \frac{dy}{y} = \int x\, dx$$

$$\ln y = \frac{x^2}{2} + C$$

$$y = e^{\frac{x^2}{2}} \cdot C \qquad \text{at } (0, 2)$$

$$= 2e^{\frac{x^2}{2}}$$

Section 2 – Calculator assumed 20 marks

Name: _____

5.   [3 marks – 2 and 1]

(a)   Continuing with $\dfrac{dy}{dx} = xy$, complete the table to find the coordinates of the next two points, starting from (2, 1), when the incremental formula (Euler's method) is applied.

$$\delta y = \dfrac{dy}{dx}\, \delta x$$

x	y	δx	δy
2	1	0.1	0.2
2.1	1.2	0.1	0.252
2.2	1.452	-	-

(b)   What can be said about the initial (boundary) condition if $\dfrac{dy}{dx} = xy$ and the graph produced using Euler's method is a horizontal line?

$$y = 0 .$$

6.   [3 marks – 2 and 1]

The velocity of particle P$_1$ is given by $v_1(t) = 2t,\ t \geq 0$ while that of P$_2$ is given by $v_2(t) = t^2 + 2t - 1,\ t \geq 0$.

Both particles are moving along the same straight line and are initially at the origin O.

(a)   When is the velocity of the two particles the same?

$$2t = t^2 + 2t - 1$$
$$\Rightarrow\ t = 1 \quad (\text{only the sol}^n)$$

(b)   What distance is covered by P$_2$ up to and when the velocities are equal?

$$\text{Distance} = \int_0^1 |t^2 + 2t - 1|\, dt = 0.77 \text{ units}$$

7. [12 marks – 2, 6, 2, 1 and 1]

An advertising executive commissioned a mathematical analysis of the effectiveness of a particular television campaign.

The rate of increase in the percentage of the market (P) aware of the product was modelled by $\dfrac{dP}{dt} = 2P - 0.025P^2$, at t weeks

(a) This equation has the rate of increase proportional to two basic quantities. What are they?

$$\frac{dP}{dt} = 0.025P(80 - P)$$

$\propto P$, current percentage

$\propto (80 - P)$ difference between a max and current P

(b) Use appropriate calculus techniques to derive $P(t) = \dfrac{2}{0.025 + C e^{-2t}}$.

Isolate Variables
$$\int \frac{dP}{P(80 - P)} = \int 0.025\, dt$$

$$\int \frac{\frac{1}{80}}{P} + \frac{\frac{1}{80}}{80 - P}\, dP = \int 0.025\, dt$$

$\times 80.$
$$\ln P - \ln(80 - P) = 2t + c$$

$$\ln\left(\frac{P}{80 - P}\right) = 2t + C$$

$$\frac{P}{80 - P} = e^{2t + C}$$

Solve for P:
$$P = \frac{80 e^{2t + C}}{1 + e^{2t + C}} \quad \div 40 e^{2t + C}$$
$$\div 40 e^{2t + C}$$

$$= \frac{2}{0.025 + C e^{-2t}}$$

If 20% of the market was initially aware of the product, determine:

(c) the proportion aware after 2 weeks of advertising

$$P(0) = 20 \Rightarrow \frac{2}{0.025 + C} = 20 \qquad \Rightarrow C = 0.075$$

$$\therefore P(2) = \frac{2}{0.025 + 0.075 e^{-4}} = \frac{75.83}{\text{?????}} \% \qquad \text{or use } e\text{-act}$$
$$\doteqdot \text{??} \% \qquad \text{on ClassPad}$$
$$76$$

(d) how long before 75% of the market is aware

$$\text{Solve} \quad \frac{2}{0.025 + 0.075 e^{-2t}} = 75 \qquad t = \overset{1.9}{3.??86} \text{ weeks}$$

(?????????????)

(e) the maximum or limiting value of market awareness

$$80\%$$

8.　[7 marks – 2, 1, 1,1 ~~2~~ and 2]

A steam-driven piston has a displacement, x, given by $x = 2\cos 2t - \sqrt{5}\sin 2t$

(a) Show that the piston is in simple harmonic motion.

$$v = -4\sin 2t - 2\sqrt{5}\cos 2t$$

$$a = -8\cos 2t + 4\sqrt{5}\sin 2t$$

$$\text{i.e. } \frac{d^2x}{dt^2} = -4\left(2\cos 2t - \sqrt{5}\sin 2t\right) = -4x$$

$$= -k^2 x \quad \text{for } k = 2$$

Determine each of:

(b) the amplitude.

$$\text{graph } x = 2\cos 2t - \sqrt{5}\sin 2t$$

$$\Rightarrow A = 3 \qquad \left(\text{or} \quad 2^2 + \sqrt{5}^2 = 9 = 3^2\right)$$

(c) the first two values of t, $t \geq 0$, when the piston is in its central (mean) position

$$t = 0.365 \quad \text{or} \quad 1.936 \quad \text{units}$$

(d) initial dir^n of travel

$$\text{from } x = +2 \text{ towards } x = 0$$
$$\left(\text{negative or downwards}\right)$$

(e)(d) the velocity when $x = \sqrt{6}$

$$v^2 = k^2\left(A^2 - x^2\right)$$

$$= 4(9 - 6)$$

$$= 12$$

$$\text{i.e. } V = \pm 2\sqrt{3} \text{ units}$$